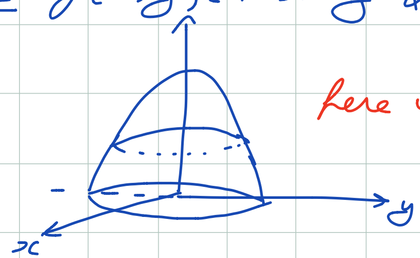
More Review

Double integrals (5.1–5.4)

$$\iint_R f(x,y)\,\underbrace{dA}_{\substack{dx\,dy \\ dy\,dx}} = \text{Volume of the region above } R \text{ and under the graph of } f \text{ when } f \text{ is non-neg.}$$

example: $f(x,y) = 1 - x^2 - y^2$ & $R: -1 \leq x \leq 1,\ -1 \leq y \leq 1$



here we're computing a "signed" volume

$$\int_{-1}^{1}\left[\int_{-1}^{1}(1-x^2-y^2)\,dy\right]dx = \int_{-1}^{1}\left(y - x^2y - y^3/3\,\Big|_{y=-1}^{1}\right)dx$$

iterated integral

$$= \int_{-1}^{1} 1 - x^2 - \tfrac{1}{3} - \left(-1 + x^2 + \tfrac{1}{3}\right)dx$$

$$= \int_{-1}^{1} \tfrac{4}{3} - 2x^2\,dx = \tfrac{4}{3}x - \tfrac{2}{3}x^3\Big|_{-1}^{1}$$

$$= \tfrac{4}{3} - \tfrac{2}{3} - \left(-\tfrac{4}{3} + \tfrac{2}{3}\right) = 4/3$$

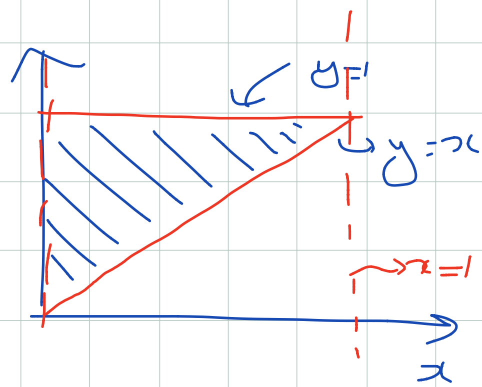
Sometimes evaluating an iterated integral can be hard so we may need to change the order of integration

Example: $\int_0^1\int_x^1 e^{y^2}\,dy\,dx$

Here, e^{y^2} doesn't have an antiderivative that we know

So we try changing the order of integration

① We sketch



so we can also define this region by $0 \le x \le y$, $0 \le y \le 1$

so $\int_0^1 \int_x^1 e^{y^2}\, dy\, dx = \int_0^1 \int_0^y e^{y^2}\, dx\, dy$

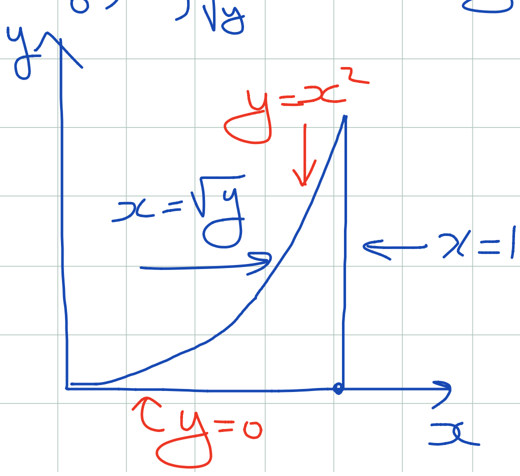
$$= \int_0^1 x e^{y^2} \Big|_{x=0}^{x=y} dx\, dy$$

$$= \int_0^1 y e^{y^2} dy$$

$$= \frac{1}{2} e^{y^2} \Big|_0^1 = \frac{e-1}{2}$$

Example: $\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dx\, dy$

Sol'n



$$\Rightarrow \int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy = \int_0^1 \int_{y=0}^{x^2} e^{x^3}\,dy\,dx$$

$$= \int_0^1 \left(y e^{x^3} \right)\Big|_{y=0}^{y=x^2} dx$$

$$= \int_0^1 x^2 e^{x^3}\,dx$$

or

(Let $u = x^3 \Rightarrow du = 3x^2 dx$)

$$= \int_0^1 e^{x^3}\,dx^3/3 = \frac{e^{x^3}}{3}\Big|_{x=0}^{x=1}$$

$$= \frac{e^1 - e^0}{3} = \frac{e-1}{3}$$

Exercise: do $\int_0^4 \int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy$

Triple Integrals (5.5)

$$\iiint_R f(x,y,z)\,\underbrace{dV}_{dA\,dz \,=\, dx\,dy\,dz}$$

R → solid in space

example: Let B be the box given by

$$0 \leq x \leq 1,\ 0 \leq y \leq 2,\ -1 \leq z \leq 0$$

evaluate $\iiint_B x^2 + xy + z^2 y\,dV$

$$= \int_0^1\int_0^2\int_{-1}^0 (x^2+xy+z^2y)\,dz\,dy\,dx \quad \text{(iterated integral)}$$

$$= \int_0^1\int_0^2 x^2z + xyz + \frac{z^3}{3}y \Big|_{z=-1}^{z=0} dy\,dx$$

$$= \int_0^1\int_0^2 x^2 + xy + \frac{1}{3}y \; dy\,dx$$

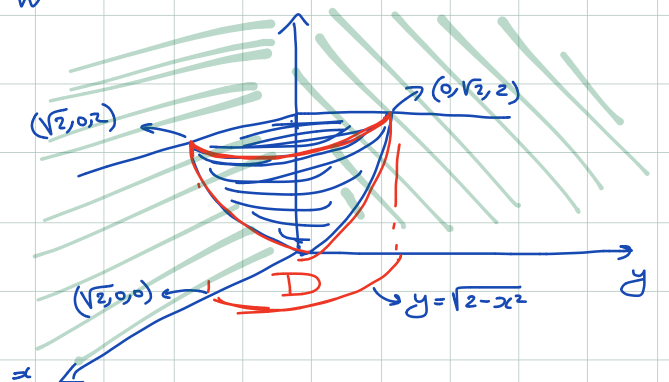
$$= \int_0^1 x^2y + xy^2/2 + y^2/6 \Big|_{y=0}^{y=2} dx$$

$$= \int_0^1 2x^2 + 2x + 2/3 \; dx = 2/3 + 1 + 2/3 = 7/3$$

<u>Exercise:</u> Verify that you get the same answer if you change the order of integration.

<u>Example:</u> Let W be the region bounded by the planes $x=0, y=0,$ $z=2$, & the surface $z=x^2+y^2$ lying in the quadrant $x\geq 0, y\geq 0$.

Compute $\iiint_W x\,dx\,dy\,dz$



So, we can say $x^2+y^2 \leq z \leq 2$, $0 \leq y \leq \sqrt{2-x^2}$, $0 \leq x \leq \sqrt{2}$

So, $\iiint_W x \, dV = \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^{2} x \; dz \, dy \, dx$

$$= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} 2x - x^3 - y^2 x \; dy \, dx$$

$$= \int_0^{\sqrt{2}} 2xy - x^3 y - \frac{y^3}{3} x \Big|_{y=0}^{y=\sqrt{2-x^2}} dx$$

$$= \int_0^{\sqrt{2}} 2x\sqrt{2-x^2} - x^3\sqrt{2-x^2} - \frac{(2-x^2)^{3/2}}{3} x \, dx$$

$$= \int_0^{\sqrt{2}} x(2-x^2)^{3/2} - x(2-x^2)^{3/2}/3 \quad dx$$

$$= \int_0^{\sqrt{2}} \frac{2}{3} x (2-x^2)^{3/2} \, dx$$

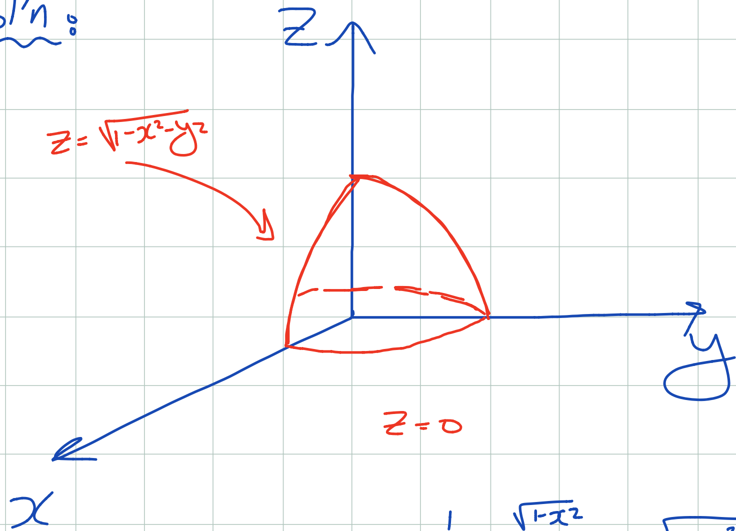
$$= \int_{x=0}^{x=\sqrt{2}} \frac{1}{3} (2-x^2)^{3/2} \, dx^2$$

$$\underset{u=2-x^2}{=} \int_2^0 -\frac{1}{3} u^{3/2} \, du = \frac{2}{5} \cdot \frac{1}{3} \, 2^{5/2} = 8\sqrt{2}/15$$

Exercise: repeat with the order of integration given by $dx \, dy \, dz$

Example: Set up the triple integral for finding the Volume & Mass of an object W bounded by the planes $z=0$, $y=0$ & the graph of $z=\sqrt{1-x^2-y^2}$, with density $f(x,y,z)$.

Sol'n:



So $\text{Volume} = \int_{x=-1}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$

$$\text{Mass} = \int_{x=-1}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

Example (part 2) Find the Mass if $f(x,y,z) = y$

$$M = \int_{-1}^{1}\int^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} y\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\;y\,dy\,dx$$ Let $y^2 = u \Rightarrow 2y\,dy = u$

$$= \int_{-1}^{1}\int_{y=0}^{\sqrt{1-x^2}} \frac{1}{2}\sqrt{1-x^2-u}\,du\,dx$$ Let $V = 1-x^2-u$, $dV = -du$

$$= \int_{-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} -\frac{1}{2} \underbrace{\sqrt{1-x^2-u}}_{V} \, d(\overbrace{1-x^2-u}^{V}) \, du$$

$$= \int_{-1}^{1} -\frac{1}{2} \cdot \frac{2}{3} (1-x^2-u)^{3/2} \Big|_{y=0}^{\sqrt{1-x^2}} \, dx$$

$$= \int_{-1}^{1} -\frac{1}{3} (1-x^2-y^2)^{3/2} \Big|_{y=0}^{\sqrt{1-x^2}} \, dx$$

$$= \int_{-1}^{1} -\frac{1}{3} \Big[\underbrace{(1-x^2-(1-x^2))^{3/2}}_{0} - (1-x^2-0)^{3/2} \Big] \, dx$$

$$= \int_{-1}^{1} +\frac{1}{3} (1-x^2)^{3/2} \, dx = \frac{\pi}{8}$$

how?

Let $x = \sin u \Rightarrow dx = \cos u \, du$

& $1-x^2 = \cos^2 u$

so $\int_{x=-1}^{1} \frac{1}{3} [\cos^2(u)]^{3/2} \cos u \, du = \int_{x=-1}^{1} \cos^4 u \, du$

$= \dots$ by parts

Aaaaah!

Turns out many problems (like this one) are much easier with a change of variables.

$\Longrightarrow$ Next Section!

Section 6.1/6.2

(Linear maps & the change of variable formula)

Example: calculate the area of the ellipse

$$\frac{x^2}{a^2} + y^2/b^2 \leq 1$$

Sol'n:

$$A(D) = \iint_{D:\, x^2/a^2 + y^2/b^2 \leq 1} dx\, dy$$

Let's make the transformation $x = u/a$, $y = v/b$

then $dx = \frac{1}{a} du$, $dy = \frac{1}{b} dv$

so $$\iint_{x^2/a^2 + y^2/b^2 \leq 1} dx\, dy = \iint_{u^2+v^2 \leq 1} ab\, du\, dv$$

$$= ab \iint_{u^2+v^2 \leq 1} du\, dv = \pi a b$$

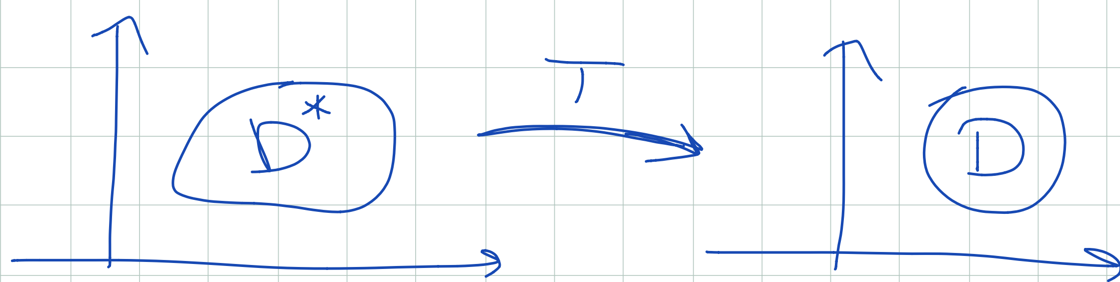
area of circle

— x —

Need a systematic way of dealing with change of variables in $\mathbb{R}^2$ or $\mathbb{R}^3$

6.1 Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

in single variable calculus, many times we needed to do a change of variable to make an integral easier.
We often need to do something similar when we have several variables. A key ingredient, in 2D is the concept of mappings between subsets of $\mathbb{R}^2$ to subsets of $\mathbb{R}^2$.



Let T be a map from $D^* \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

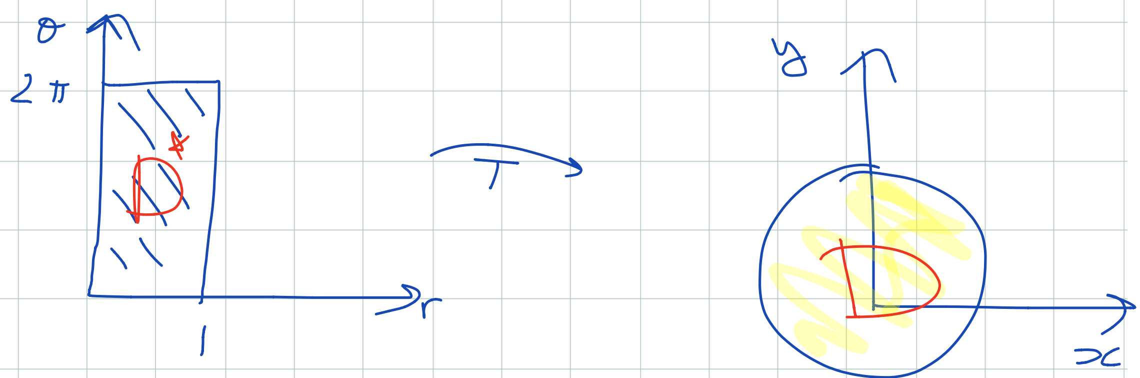
We call $D = T(D^*)$ the set of image points of T

(So every pt (x,y) in D must be equal to $T(x^*,y^*)$ for some (x^*,y^*) in D^*)

Example: (Polar coordinates)

Let D^* be defined by $0 \leq r \leq 1$ & $0 \leq \theta \leq 2\pi$

$$T(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$$



because $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2 \leq 1$

so $D = T(D^*) \subset$ Ball of radius one.

But is it the whole Ball? Yes! Because for any pt (x,y) such that $x^2+y^2 \leq 1$

there is an (r,θ) so that $x = r\cos\theta$, $y = r\cos\theta$.

<u>Theorem</u>: Let A be a 2×2 matrix with $\mathrm{Det}(A) \neq 0$

(Reminder/Note: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\mathrm{Det}(A) = ad - bc$).

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(\vec{x}) = A\vec{x}$

(In otherwords $T(x,y) = (ax+by, cx+dy)$).

Then T transforms parallelograms to parallelograms & vertices to vertices. Moreover, if $T(D^*)$ is a parallelogram, then D^* is a parallelogram.

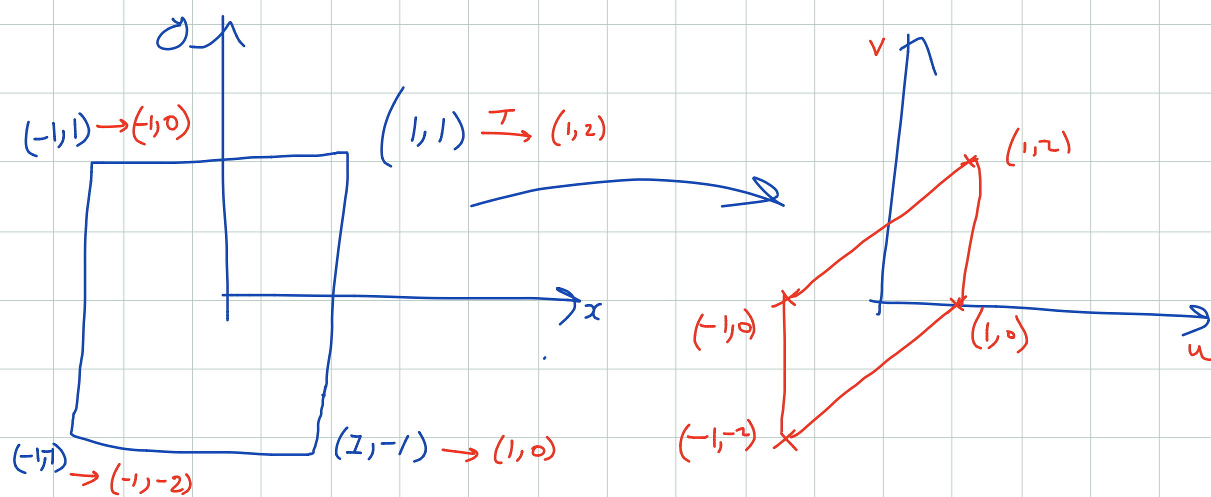
Example: Let $T(x,y) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x+y \end{pmatrix}$

and let $D^* = [-1\ \ 1] \times [-1\ \ 1] \subset \mathbb{R}^2$

(this just means $-1 \leq x \leq 1$ & $-1 \leq y \leq 1$)

find & sketch $D = T(D^*)$

Solution: $\mathrm{Det}\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right) = 1 \times 1 - 0 = 1$ so by the theorem, we only need to map the vertices (and connect them)



So $T(D^*)$ is the pictured parallelogram.

6.2 The Change of Variable Theorem

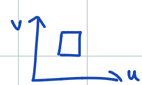
$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} \boxed{?}\, du\,dv$ if we do a change of variable

(e.g. Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$)

So $T(r,\theta) = (r\cos\theta, r\sin\theta)$)

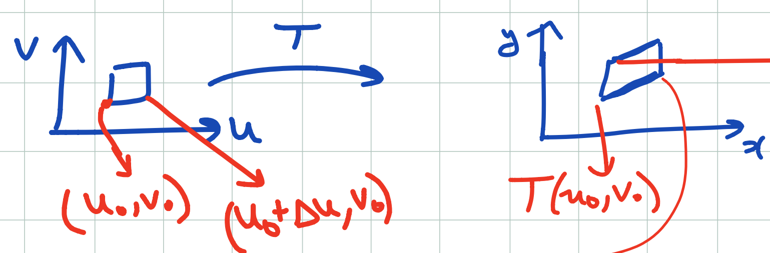
Need a systematic way of doing this for general change of Var.

Let's consider a small rectangle D^* in the uv-plane.



and let $T'_{(u_0,v_0)}$ be the derivative of $T: D^* \to D$ (so T' is 2×2) evaluated at (u_0, v_0)

Linear approximation: $\boxed{T(u,v) \approx T(u_0,v_0) + T'\big|_{(u_0,v_0)} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}}$ ($\Delta u \to u-u_0$, $\Delta v \to v-v_0$)



$T(u_0,v_0) + \begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{bmatrix}_{(u_0,v_0)} \Delta v$

$T(u_0,v_0) + \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}_{(u_0,v_0)} \begin{bmatrix} \Delta u \\ 0 \end{bmatrix} = T(u_0,v_0) + \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{bmatrix}_{(u_0,v_0)} \Delta u$

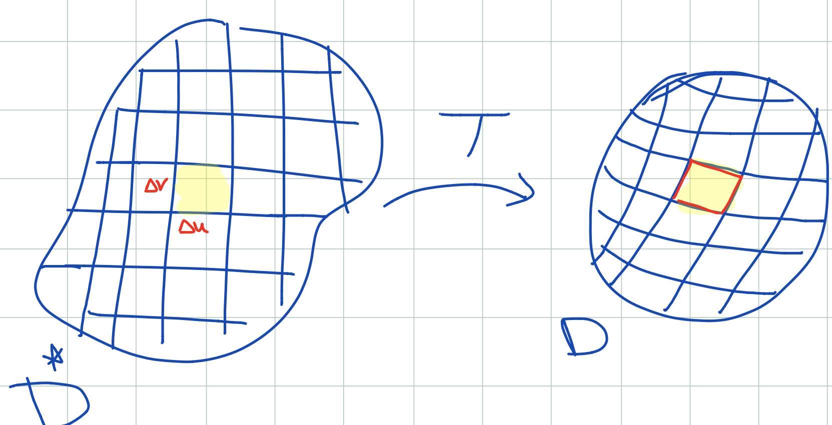
So the area of the parallelogram is equal to the area of a p'gram with sides $\frac{\partial x}{\partial u}\Delta u\,\vec{i} + \frac{\partial y}{\partial u}\Delta u\,\vec{j}$ and $\frac{\partial x}{\partial v}\Delta v\,\vec{i} + \frac{\partial y}{\partial v}\Delta v\,\vec{j}$

Area of the p'gram $= \left|\begin{matrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v \end{matrix}\right|$

$$= \left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right]\Delta u \Delta v$$

$$= \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right| \Delta u \Delta v$$

Let's now do this for the tiny squares that make up D^*



$A(D) =$ sum of the areas of the parallelograms

$$= \sum_{\text{all }(u_0,v_0)}\sum \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right|_{(u_0,v_0)} \Delta u \Delta v \xrightarrow[\text{shrink}]{\text{as rect.}} \iint_{D^*} \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right| du\,dv$$

So $A(D) = \int_{D^*} \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right| du\,dv$